The Journal of Financial Engineering ¾ *Volume 5* ¾ *Number 3* ¾ *Pages 211-227.*

Simulation and the Early-Exercise Option Problem

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ABSTRACT

This article uses Monte Carlo simulation to identify optimal early-exercise condition(s) for options. Thus, Monte Carlo simulation can value American-style options for which there is no closed-form solution and which may be too complex for other numerical methods. We first illustrate the procedure and demonstrate its accuracy by valuing an option with a known solution, the American put on an asset price that follows a pure diffusion stochastic process. We then demonstrate the flexibility of the method and its capacity to value options that other methods cannot, by valuing an American put on an asset price that follows a jump-diffusion stochastic process.

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We have benefited from conversations with Jerry Baesel, Phelim Boyle, Bob Whaley, John Hull, and our colleagues at the University. The thoughtful comments and suggestions of anonymous referees have improved the paper. We also thank the participants in Options Seminar (April 1993) sponsored jointly by the Department of Finance and the Office of Futures Markets, University of Illinois, for their helpful comments. We alone, however, are responsible for any errors.

I. INTRODUCTION

Black and Scholes (1973) developed a closed-form solution for the value of a call option with relatively simple features. Closed-form valuation solutions are precise and efficient. While these virtues are undeniable, closed-form solutions for the value of options with more complex characteristics either do not exist or have not yet been developed. As a result, much of the recent options research focuses on the use of numerical methods to calculate the values of complex options. This article expands the numerical methods available to value complex options, by using Monte Carlo simulation to value options with early-exercise features, *i.e.*, American-style options.

The classic example of the early-exercise feature is found in the American put. Early exercise is advantageous when the American put is sufficiently in-the-money. The early-exercise feature of the American put has kept researchers from developing a closed-form valuation solution for it. There are, however, several numerical methods that can value the American put. Parkinson (1977), for example, presented a numerical integration method. Brennan and Schwartz (1977) proposed the finite differences method and demonstrated its effectiveness. Sharpe (1978) introduced the idea of valuing options by limiting the distribution of future prices of the underlying asset to two points. Cox, Ross, and Rubinstein (1979) extended this idea to develop what has come to be known as the binomial tree or lattice method of option valuation.[1] Johnson (1983) and Geske and Johnson (1984) developed solutions for the American put by treating the opportunity to exercise early as an "option" to exercise early (the compound option method). Lastly, MacMillan (1986) and Barone-Adesi and Whaley (1987) developed a fast, simple analytic approximation of the value of an American put.

Boyle (1977) introduced Monte Carlo simulation as another numerical method to use to value options. While he demonstrated the use of simulation to value European-style options, until recently Boyle has been interpreted to mean that simulation cannot be used to value American-style options, including American puts, because of the early-exercise feature. For example, Hull observes, “One limitation of the Monte Carlo approach is that it can be used only for European-style derivative securities.”[2] Hull and White make the reasoning clear when they write, “Monte Carlo simulation cannot handle early exercise since there is no way of knowing whether early exercise is optimal when a particular price is reached at a particular time.”[3]

Tilley (1993) was the first to publish a procedure for incorporating early exercise in a Monte Carlo simulation. His approach requires storage of the paths followed by the asset price, their ranking, and their re-ranking at each possible early-exercise date. Tilley illustrates his approach for a standard American put. By grouping the ranked asset prices at each date, he is able to estimate for the individual groups of asset prices whether early exercise is optimal for that group at that date. Tilley reports, but does not demonstrate, that it is possible to extend his procedure to incorporate two or more influences on the early-exercise decision. It appears that the ranking process and its associated heuristic rules make this method much more complex and much more time-consuming than the method we present.

Recently, Barraquand and Martineau (1995) also proposed an approach that tracks the conditional probabilities of path-specific outcomes in a Monte Carlo simulation. They use these values to make early-exercise decisions and to value American puts written on the maximum of 1, 3, and 10 underlying homogenous asset prices. Writing the put on the maximum of multiple assets, rather than the minimum, reduces the relevant state space rapidly when the initial asset prices are close to the exercise price, *i.e.,* the put is almost certain to be out-of-the-money in the future because at least one of the many asset prices will exceed the exercise price. This simplifies the problem because it narrows the state space that the authors must examine.

In this article, we demonstrate how to simply and directly determine, “whether early exercise is optimal when a particular price is reached at a particular time.” This allows us to incorporate the early-exercise feature into Monte Carlo simulation in such a way that it is readily extendible to very complex options. We first apply the solution procedure by valuing an American put because this enables us to compare our solution with known values. This comparison highlights the accuracy of the procedure. We then value an American put on a stock whose price follows a jump-diffusion stochastic process. This illustrates an application where there is no known alternative valuation method.

Extending the use of Monte Carlo simulation to value American-style options is important because the Monte Carlo simulation is an inherently flexible method. It can readily accommodate many complexities. Among the more important complexities are a stochastic interest rate, multiple stochastic processes, multiple underlying assets, path-dependence, and customized features of exotic options. Many of these complexities are compounded by the early-exercise provision, and other numerical methods are not capable of valuing the options. Therefore, this demonstration may establish Monte Carlo simulation as the valuation method of choice for complex American-style options.

In Section II, we describe the implementation of Monte Carlo simulation to value an American-style security. In Section III, we provide a numerical illustration of the method for an American put on a stock whose price follows a pure diffusion stochastic process. In Section IV, we illustrate the method by valuing an American put on a stock whose price follows a jump-diffusion process. Section V relates this article to the evolving literature, and offers concluding remarks.

II. MONTE CARLO SIMULATION FOR AMERICAN-STYLE OPTIONS

To value American-style options, we must be able to identify, at each date, the optimal early-exercise decision, which depends on optimal early exercises at all future dates. Therefore, we must identify the optimal early-exercise criteria using the backward recursive technique of dynamic programming. Monte Carlo simulation has not been applied to American-style options because of a seeming inability to link forward-moving Monte Carlo simulation to backward-moving dynamic programming. In terms of the American put, the challenge is to identify at each date what Merton (1973) refers to as the *critical price* of the underlying asset, *i.e.*, the price below which it is optimal to exercise early.

The rationality condition requires the early exercise of the put, if the current payoff from exercise is greater than the expected present value of the payoff from holding the put. Implementation of the rationality condition requires the ability to estimate the expected present value of the payoff from holding the put. This, in turn, depends upon knowledge of the critical prices at future dates. We break the circularity of this problem by recognizing that we can initiate a Monte Carlo simulation at any date and for any price of the underlying asset. Therefore, we can estimate the expected present value of holding the put at any date and for any price of the underlying asset, given knowledge of all future critical prices. By initiating Monte Carlo simulations for each stock price in a large enough range of prices less than the exercise price, we can identify the prices for which the expected present values of holding the put are higher and lower than the payoff from exercise, *i.e.*, we can identify the critical price of the underlying asset at that date.[4] The critical price of the underlying asset at the expiration date is simply the exercise price, X. Because we know this terminal condition, we can use backward-recursive dynamic programming to estimate the critical prices, beginning with the last date prior to expiration.

As noted earlier, Hull and White expressed the widely held view that it was not possible to know the optimality of early exercise in a forward moving simulation. This view appears to arise from thinking in terms of initiating Monte Carlo simulations only from the valuation date of the option.[5] Such a simulation is necessary to estimate the value of the put, but it is not necessary to answer other questions, such as, "What is the critical price of the underlying asset one month before expiration (regardless of the initial price of the underlying asset)?"

We illustrate the procedure by valuing an American put at date 0, with an expiration date of T. The underlying asset pays no dividends and its price, S, follows the standard geometric brownian motion stochastic process given by:

$$dS/S = a\ dt + s\ dz \tag{1}$$

where a and s are the instantaneous rate of return and instantaneous standard deviation of the rate of return, respectively, and z is a unit normal random variable. The instantaneous risk-free rate of interest is r. Employing the principle of risk-neutral valuation (Cox and Ross (1976)), we can substitute $(r - s^2/2)$ for a.

At any date t, the value of the American put is P_t. To calculate the value of the put, it is necessary to identify the critical prices, S_t^*, at all dates between t and T. We can state the valuation problem formally as:

$$P_0(S_t^*) = \max_{\{S_t\}} Q_t[S_t] \tag{2}$$

where $Q_t = X - S_t$ when early exercise is optimal, *i.e.*, $S_t < S_t^*$, or $Q_t = E[P_{t+t}(S_{t+t}^*)]e^{-rt}$ when holding is optimal, *i.e.*, $S_t \geq S_t^*$. E is the expectations operator and t is an arbitrarily small unit of time, expressed as a fraction of a year.

Because the early-exercise decision at each date depends on the knowledge of the optimal early-exercise decisions at all future dates, we must employ the backward recursion of dynamic programming, beginning with the terminal condition. At date T, it is optimal to exercise if the put is in-the-money; then its value is $P_T = \max(0, X - S_T)$, *i.e.*, $S_T^* = X$. The optimization process begins at the last date before the put expires, $T - t$. The holder of the put can exercise early or hold until the expiration of the put. At date $T - t$, the value of the put is $X - S_{T-t}$ for $S_{T-t} < S_{T-t}^*$, or $E[P_T]e^{-rt}$ for $S_{T-t} \geq S_{T-t}^*$. We identify the critical price by finding the price for which $X - S_{T-t}^* = E[P_T]e^{-rt}$. Assuming it is possible to identify S_{T-t}^*, the optimization continues by identifying S_{T-2t}^* at date $T - 2t$, conditional on the knowledge of S_{T-t}^*. This process continues at $T - 3t$, and back to date 0.

Solution of this optimization problem requires identifying S_t^*, by comparing $X - S_t$ and $E[P_{t+t}]e^{-rt}$, conditional on the knowledge of all of the critical prices at dates later than t. The problem is that $E[P_{t+t}]e^{-rt}$ depends on information about future prices that the simulation has not produced at date t. We solve this problem by initiating a Monte Carlo simulation at *a particular date, for any chosen initial condition*. In terms of the problem at hand, this means we initiate a Monte Carlo simulation at date $T - t$, with an initial condition S_{T-t}, that generates values of S_T, as:[6]

$$S_T = S_{T-t} e^{(r - s^2/2)(t) + s\sqrt{t}\, z} \tag{3}$$

Having generated S_T, we estimate $E[P_T] = E[\max(0, X - S_T)]e^{-rt}$ by averaging across the simulation results for $\max(0, X - S_T)e^{-rt}$. We compare $E[P_T]$ with $X - S_{T-t}$. This establishes whether the critical price is above or below S_{T-t}. By beginning with S_{T-t} arbitrarily close to the exercise price, and repeating this process for a series of prices that decline systematically, we identify the critical price S^*_{T-t}, where $(X - S^*_{T-t}) = E[P_T]$.

We can express this Monte Carlo simulation solution formally as:

$$P_t(S_t^*) = \max_{\{S_t\}}\{(1/N)\sum_{i=1}^{N} Q_{t,i}[S_t]\} \tag{4}$$

where i is the simulation run. The solution begins with Monte Carlo simulations of S_T, for a range of $S_{T-t} < X$ that is large enough to identify S^*_{T-t}. This permits estimation of the expected present value of holding the put, for each initial price. We compare these values with the payoff from exercise of the put, $X - S_{T-t}$, to identify the early-exercise range as all prices for which $X - S_{T-t} > E[P_T]e^{-rt}$. The solution process continues, stepping backward in time for the entire life of the option. Once the critical price at each date has been identified, we estimate the value of the option through a simulation initiated at time 0, for the appropriate initial conditions. Early exercise occurs on the first date that the stock price falls below the critical price.

This theoretical solution calls for an infinite set of early-exercise dates. In practice, it is necessary to identify only as many critical prices as required to achieve an acceptable approximation of the value of the option. In the next section, we illustrate the solution and indicate its potential for accurate estimation.

III. VALUING AN AMERICAN PUT: THE PURE DIFFUSION CASE

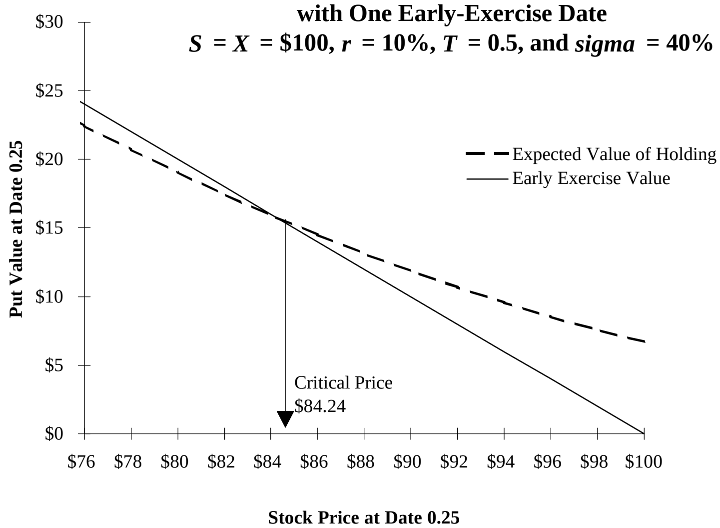
In this section, we use Monte Carlo simulation to value an American put on a common stock whose price follows a pure diffusion process. Let $S_0 = X = \$100$, $T = 0.5$ years, $r = 0.10$, and $s = 40\%$.[7] First we estimate critical prices, incorporating the backward recursion of dynamic programming,[8] and estimate the value of the put. We then investigate the accuracy required in the estimates of the critical prices, the number of early-exercise dates required for accurate estimates of the put value, and the sensitivity of the results to changes in the parameters of the put. Further, we discuss the computer time required by this procedure.

Estimating Critical Prices and Valuing the Put

The estimation of critical prices is best explained by allowing early exercise at only one point, halfway through the life of the put, at $t = 0.25$. To value the put we must know the critical price at date 0.25. We initiate a set of 13 simulations, each with a different initial price, over the range \$100, \$98, \$96, ..., \$78, \$76.[9] For each simulation we estimate the expected value of holding the put. Figure 1 plots the results of these simulations and the payoff from early exercise. The expected present value of holding the put ranges from \$6.72, when the stock price is \$100, to \$22.44, when the stock price is \$76. In the former case, early exercise is irrational because the payoff is \$0.00. In the latter case, early exercise is rational because the payoff is \$24.00, or \$1.56 more than the expected present value of holding. When the initial price is \$86, the early-exercise payoff is \$14.00, the expected present value of holding is \$14.49, and it is not optimal to exercise early. When the initial price is \$84, the early-exercise payoff is \$16.00, the expected present value of holding is \$15.93, and it is optimal to exercise early. Therefore, the critical price falls between these two prices. To estimate the critical price, we assume that between \$86 and \$84 the expected value of holding is a linear function of the initial price. We know that the payoff from early exercise is also a linear function of the initial price. Therefore, it is possible to estimate the initial price for which the two values are equal.

In Figure 1, the plots of payoffs from early exercise and the expected present value of holding cross at approximately \$84.24. *This is the critical price*. For stock prices above \$84.24 it is optimal to hold the put, but for prices below \$84.24 it is optimal to exercise early. We estimate the value of the put by initiating a Monte Carlo simulation at time 0, with S_0 = \$100, by simulating prices at date 0.25 and by holding or exercising at date 0.25, depending upon whether the prices are above or below the critical price. The estimate is \$8.95. This compares with the value of the corresponding European and American puts of \$8.70 and \$9.22, respectively.[10]

Figure 1
Pure Diffusion Put Values for Exercising Early and Holding with One Early-Exercise Date
***S* = *X* = \$100, *r* = 10%, *T* = 0.5, and *sigma* = 40%**



Incorporating Backward Recursion for More Than One Early-Exercise Date

To illustrate the backward recursive application of the simulation process, we permit early exercise at two evenly spaced dates, $t = 0.1667$ and $t = 0.3333$.[11] We estimate the critical price at date 0.3333 exactly as described in the preceding paragraph. It is $85.65. Note that the critical price is higher at date 0.3333 than at date 0.25, *i.e.*, the critical price rises as the time-to-expiration declines. To estimate the critical price at date 0.1667, we initiate a set of Monte Carlo simulations at that date, each with a different initial price over the range $100, $98, $96, ..., $78, $76. For each simulation we estimate the expected value of holding the put. Within a simulation, each iteration calculates a price at date 0.3333 and determines whether to exercise by comparison with the critical price, $85.65. If exercise is indicated, the value of the put for that iteration is its present value from early exercise. If early exercise is not indicated, the simulation continues for that iteration by generating a price at expiration, at date 0.50.

The value of the put for such an iteration is the present value of its terminal value. For each initial price at date 0.1667, this process estimates an expected present value of the put at date 0.1667, conditional on optimal exercise at date 0.3333. By comparing this value with the value of exercise at date 0.1667, we can identify the critical price at date 0.1667. It is $81.08. With the critical prices at dates 0.1667 and 0.3333 now available, it is possible to estimate the value of the put by initiating a simulation at date 0 with early-exercise dates of 0.1667 and 0.3333. This produces an estimate for the value of the put of $9.05, as compared with a value of $8.95 for only one early-exercise date. This process can be repeated for as many early-exercise dates as required by the application.

Accuracy of the Critical Price Estimates

The Monte Carlo simulation technique estimates critical prices with errors. These errors bias downward the estimate of the value of the put. We can reduce these errors by increasing the number of iterations used to estimate the critical prices. Therefore, the magnitude of this bias is an experimental issue, with implications for the efficiency of the procedure.

To identify an appropriate number of iterations, we need two pieces of information. The first is the error in the estimate of the critical price induced by the different numbers of iterations used. The second is the sensitivity of the estimate of the put value to errors in the estimate of the critical price. We examine these issues in the context of the example used earlier, namely, the put with a single early-exercise date.

To measure the potential error in estimates of the critical price, we estimate it 50 times for each of four Monte Carlo simulations with 1,000; 2,000; 3,000; and 4,000 iterations, respectively. Table 1 summarizes the results of the simulations. Based on these values, which are consistent with the result reported in Figure 1, it appears that when we use as few as 1,000 iterations to estimate the critical price, it falls within $0.40 of the true value with a 95% probability. Therefore, we examine the estimates of the put value for five critical prices, $84.70; $84.50; $84.30; $84.10; and $83.90.

The data in Figure 1 suggest that $84.30 is an accurate estimate of the true critical price. The remaining prices represent deviations of $0.20 and $0.40 from that estimate.[12] These are relatively large deviations, compared to the estimates of the likely errors. The estimated values of the put for these critical prices are $8.9557; $8.9563; $8.9577; $8.9565; and $8.9550; respectively. The errors introduced by relatively large errors in the estimate of the critical price appear to be small fractions of one cent. Therefore, the put estimate does not appear to be very sensitive to errors in the estimates of the critical prices. These results suggest that, with as few as 1,000 iterations, it is possible to estimate critical prices that impart relatively little bias to the estimate of the value of the put.[13]

Table 1
Estimates of the Critical Price Based on 50 Repetitions
***X* = $100, *S* = $100, *r* = 10%, *T* = 0.5, *V* = 40%, One Early-Exercise Date**

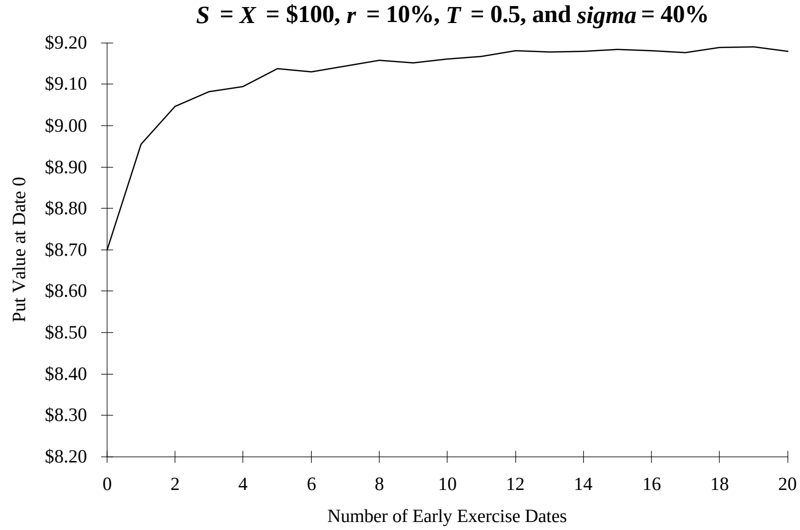
Iterations	1,000	2,000	3,000	4,000
Mean	$ 84.25	$ 84.31	$ 84.32	$ 84.30
Standard Deviation	$ 0.20	$ 0.15	$ 0.10	$ 0.12

Number of Early-Exercise Dates

A second source of bias is introduced when we restrict the number of early-exercise dates. If we take as our benchmark a theoretical model in which investors can exercise the put at any instant, then there is a theoretically infinite number of early-exercise dates. A practical simulation solution allows early exercise at a relatively small number of dates.

Figure 2 reports the estimated values of the put for 0 to 20 evenly spaced early-exercise dates.[14] The additions of the first few early-exercise dates have substantial effects, but additional early-exercise dates have much smaller effects. Over the range from 12 to 20 early-exercise dates, the estimated values of the put are in the range $9.18 to $9.19, with a standard error of estimate of $0.01. These values are 99.6% to 99.7% of the lattice estimate of $9.22. Thus, we are able to closely estimate the value of the put, with relatively few early-exercise dates.

Figure 2
Pure Diffusion Put Values with Evenly Spaced Early-Exercise Dates
***S* = *X* = $100, *r* = 10%, *T* = 0.5, and *sigma* = 40%**



Sensitivity to Parametric Changes

To determine whether the estimates of the put are sensitive to the choice of parameters, we investigate the effects on the estimates of changing the parameters of the put. For 20 early-exercise dates, each of the four individual changes in the parameter values, *s* = 20%, *S* = $90, *S* = $110, and *T* = 0.25, produced Monte Carlo simulation estimates that are 99.2%, 99.3%, 99.5%, and 98.1%, respectively, of the corresponding lattice estimates. For these data it appears that only the time-to-expiration parameter has even a modest effect on the accuracy. These results may not be generalizable, and when applying this technique to other options, it is desirable to investigate the questions of sensitivity for the specific options valued.

Estimation Speed

The last issue we address is the speed of estimation. For this experiment we used a FORTRAN program running on a Pentium 100 PC. With 20 early-exercise dates, 1,000 iterations to estimate the critical prices, and 200,000 iterations to estimate the put value with a standard error of $0.01, the elapsed time is 2 minutes and 10 seconds. While not exactly instantaneous, this extension of Monte Carlo simulation to American option valuation holds the promise of providing rapid solutions that would be useful in many complex decision-making scenarios.

IV. VALUING AN AMERICAN PUT: THE JUMP-DIFFUSION CASE

Merton (1976) proposed a model which includes both geometric Brownian motion and random stock price jumps, and derived expressions for the values of European calls and puts in this jump-diffusion model.[15] The jump-diffusion process is attractive because it permits price discontinuities. Price discontinuities are a moderately common feature of stock prices generated by the random arrival of significant new information affecting the prices. The presence of random jumps of random sizes nevertheless complicates the valuation of the American put and we are not aware of any method of valuing it. It does not appear possible, for example, to incorporate random jumps in a recombining lattice. Only Monte Carlo simulation appears capable of effectively incorporating the jump process. Therefore, this is an appropriate application of our valuation method.

In Merton’s model, stock prices evolve according to a mixed stochastic process given by:

$$dS/S = (a - lk)dt + s\,dz + dq \qquad (5)$$

The variables a, s, and dz retain their earlier definitions. The expression dq captures the effect of the jump, and dz and dq are independent. The number of jumps is a Poisson-distributed variable with frequency l. The size of each jump is independently, log-normally distributed with mean m and variance d^2, and $k = [\exp(m + d^2/2) - 1]$. The usual practice is to eliminate the effect of the jump on the drift term by letting $m = -d^2/2$, *i.e.,* $k = 0$. If the jumps are diversifiable, then risk-neutral valuation of options is applicable, and we can represent the price evolution as:

$$S_{t+t} = S_t e^{(r - s^2/2)t + s\sqrt{t}\, z_0 + \sum_{i=1}^{q} d\sqrt{t}\, z_i} \qquad (6)$$

where z_0 is the unit normal random variable associated with the diffusion process, q is the number of jumps as determined by a drawing from a Poisson distribution with frequency lt, and the z_i are the independent unit normal random variates that determine the size of each jump. Note that when $q = 0$, there is no jump component to the price evolution.

To facilitate comparison with the pure diffusion example, we use the same parameters, $S = X = \$100$, $T = 0.5$, and $r = 10\%$. We also want the instantaneous standard deviation of the jump-diffusion process, $[ld^2 + s^2]^{1/2}$, to be 40%. To achieve that we set the frequency of the jumps, l, equal to 2; the instantaneous standard deviation of jumps, d, equal to 20%; and the instantaneous standard deviation of the diffusion process, s, equal to 28.28%.[16]

In Figure 3, we plot the values of the put as a function of the number of early-exercise dates. With zero early-exercise dates, the estimated value of the put is $8.39 as compared with $8.70 for the pure diffusion case, a difference of 3.7%. This is consistent with Merton's valuation of European-style puts. The put approaches a maximum value of approximately $8.81 as the number of early-exercise dates increases. This is 4.3% less than the value of $9.19 for the pure diffusion case, *i.e.*, the pure diffusion and jump-diffusion price difference for the American put is somewhat larger than for the European put. In addition to values of the puts differing for the two examples, the critical prices differ. In Figure 4 we plot the critical prices for the pure diffusion and the jump diffusion cases. While the prices follow similar patterns, the jump-diffusion critical prices are uniformly larger than the pure diffusion critical prices.

Figure 3
Jump-Diffusion Put Values with Evenly Spaced Early-Exercise Dates
***S* = *X* = \$100, *r* = 10%, *T* = 0.5, *sigma* = 28%,**
lamda = 2 and gamma = 20%

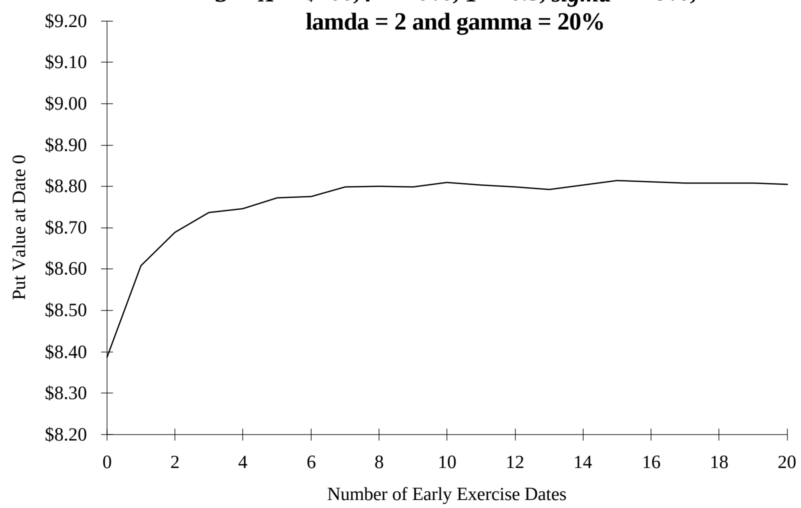
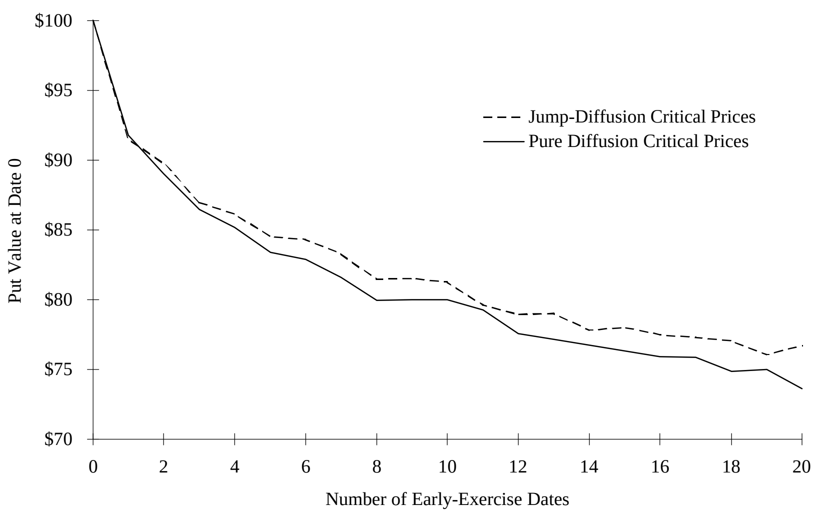


Figure 4
Critical Prices with 20 Evenly Spaced Early-Exercise Dates



V. CONCLUSION

Contrary to received wisdom, it is possible to value American-style options using Monte Carlo simulation. We do so for the familiar case of a standard American put. We also value an asset for which there is no existing valuation method: an American put whose price follows a jump-diffusion process. This demonstrates the versatility of Monte Carlo simulation with respect to complex stochastic processes. The other important type of complexity the method can accommodate is multiple influences on the early-exercise decision. Consider, for example, an American put written on the lower of two asset prices. Valuing this option requires identification of the *critical locus* of asset prices at each date. This can be accomplished with a two-dimensional grid search.[17] Similarly, an American put on the lowest of three asset prices requires a three-dimensional grid search to identify the *critical plane* of asset prices.

This article applies Monte Carlo simulation to value American puts on assets with price distributions of increasing complexity—-first, the lognormal and then the jump-diffusion. Researchers and practitioners are likely to identify numerous and novel applications of this method of valuing American-style options. With improvements in sampling methods and other variance reduction techniques, and the rapid increase in computing power, the Monte Carlo method should become increasingly useful, even for applications where time is crucial.

REFERENCES

Ball, C. and W. Torous, “On Jumps in Common Stock Prices and Their Impact on Call Pricing,” *Journal of Finance*, 40, 1985, pp. 155-73.

Barone-Adesi, G. and R. E. Whaley, “Efficient Analytic Approximations of American Option Values,” *Journal of Finance*, 42, 1987, pp. 301-320.

Barraquand, J. and D. Marineau, “Numerical Valuation of High Dimensional Multivariate American Securities,” *Journal of Financial and Quantitative Analysis*, 30, 1995, pp. 383-405.

Black, F. and M. S. Scholes, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, 81, 1973, pp. 637-654.

Boyle, P., **“**Options: A Monte Carlo Approach,” *Journal of Financial Economics*, 4, 1977, pp. 323-38.

Boyle, P., “A Lattice Framework for Option Pricing with Two State Variables,” *Journal of Financial and Quantitative Analysis*, 23, 1988, pp. 1-12.

Breen, R., “The Accelerated Binomial Option Pricing Model,” *Journal of Financial and Quantitative Analysis*, 26, 1991, pp. 153-164.

Brennan, M. J. and E. S. Schwartz, **“**The Valuation of American Put Options,” *Journal of Finance*, 32, 1977, pp. 449-62.

Cox, J. C. and S. A. Ross, “The Valuation of Options for Alternative Stochastic Processes,” *Journal of Financial Economics*, 3, 1976, pp. 145-166.

Cox, J. C., S. A. Ross, and M. Rubinstein, “Option Pricing: A Simplified Approach,” *Journal of Financial Economics*, 7, 1979, pp. 229-263.

Geske, R. and H. E. Johnson, “The American Put Valued Analytically,” *Journal of Finance*, 39, 1984, pp. 1511-24.

Grant, D., G. Vora, and D. Weeks, “Path Dependent Options: Extending the Monte Carlo Simulation Approach,” Working Paper, University of New Mexico, 1992, revised 1995.

Hull, J. *Options, Futures, and other Derivative Securities,* 2e, Englewood Cliffs, NJ: Prentice Hall, 1993.

Hull, J. and A. White, “Efficient Procedures for Valuing European and American Path-Dependent Options,” *Journal of Derivatives*, 1, 1993, pp. 21–31.

Johnson, H. E., “An Analytical Approximation for the American Put Price,” *Journal of Financial and Quantitative Analysis*, 18, 1983, pp. 141- 48.

MacMillan, L. W., “Analytic Approximation for the American Put Option,” *Advances in Futures and Options Research*, 1, 1986, pp. 119-39.

Merton, R. C., “Theory of Rational Option Pricing,” *Bell Journal of Economics and Management Science*, 4, 1973, pp. 141–183.

Merton, R. C., “Option Pricing When Underlying Stock Returns Are Discontinuous,” *Journal of Financial Economics*, 3, 1976, pp. 125-44.

Mood, A. M., F. A. Graybill, and D. C. Boes, *Introduction to the Theory of Statistics* 3e, New York, NY: McGraw-Hill, 1974.

Parkinson, M., “Option Pricing: The American Put,” *Journal of Business*, 1977, pp. 21-36.

Sharpe, W. F., *Investments*, Englewood Cliffs, N.J.: Prentice Hall, 1978.

Tan, S. T. and K. R. Vetzal, “Early Exercise Regions for Exotic Options,” *Journal of Derivatives*, 3, 1995, pp. 42-56.

Tilley, J. A., “Valuing American Options in a Path Simulation Model,” *Transactions of the Society of Actuaries*, 45, 1993, pp. 83–104.

NOTES

[1]Breen (1991) has extended this application by developing an accelerated binomial pricing model.

[2]Hull (1993) p. 334.

[3]Hull and White (1993) p. 1.

[4]In this paper, we value the American put. This requires the identification of a unidimensional early-exercise price at each date. The method is capable of incorporating multidimensional early-exercise criteria, as is required with more complex options.

[5]The solution proposed by Tilley adheres to this perspective.

[6]Note that this is equivalent to sampling from a lognormal distribution of asset price-relatives with the mean $\exp(rt)$**Error! Main Document Only.**, and the variance $\exp(2rt)\{\exp(s^{2}t) - 1\}$**Error! Main Document Only.**. See Mood, Graybill, and Boes (1974), pp. 107-119, and Hull (1993), pp. 210-14.

[7]The mean of the risk-neutral equivalent stochastic process is the risk-free rate of return minus one-half the variance of the process. In this case it is 0.10 - 0.16/2 = 0.02 per annum.

[8]For a related discussion of the identification of critical prices, see Tan and Vetzal (1995).

[9]This is just one simple search technique. In this example, we use 1,000 iterations to estimate the expected present value of the put. We justify this choice later in the paper.

[10]The American put value is the average of values generated by lattices with 500 and 501 steps. It is not surprising that the simulation estimate of $8.95 is lower than the lattice estimate of $9.22 because the simulation includes only one early-exercise date, whereas the lattice includes 500 early-exercise dates.

[11]This spacing is for illustration only. Optimal spacing of a limited number of early-exercise dates is an interesting question in its own right.

[12]Table 1 summarizes the results of 500,000 estimates of the critical price. The average of these is approximately $84.30. With as few as 1,000 iterations, we obtain a high degree of accuracy.

[13]It is important to note that this conclusion may not be valid for other options.

[14]For these estimates we use 1,000 iterations to estimate each critical price and 200,000 iterations to estimate the value of the put. The standard error of all estimates is $0.01.

[15]See Hull (1993) p. 443 and Ball and Torous (1985).

[16] $.4^2 = ld^2 + s^2$ and $s = \sqrt{.16 - 2(.2)^2} = .2828427$.

[17]Grant, Vora, and Weeks (1995) provide a detailed description of how to implement the method when two parameters influence the early-exercise decision. They value American-style Asian options by using a two-dimensional grid search, at each early-exercise date, to identify the critical locus of the asset price and the average asset price.