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Valuing Embedded Derivatives in Equity Tranche Financing[1]

by

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[1] Not for quotation. Please do not reference without permission. He is a Senior Associate at PwC. We completed this research when we were colleagues there.

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Introduction

This paper illustrates how to value the equity and derivatives created in multi-tranche equity financing. As an example, consider a firm that enters into a contract with investors to issue 10 million shares of stock at the closing of the contract at $1.00 per share (tranche 1) and to issue 10 million more shares in one year and 10 million more in two years (tranches 2 and 3), also at $1.00 per share. The issuance of tranches 2 and 3 can be certain or contingent on future events, such as the achievement of business targets. When the issuance of future tranches is certain, the initial contract is a sale of shares and forward contracts on the shares. When the issuance of future tranches is contingent, the initial contract is a sale of shares and call options on the shares.

Tranche Financing with Certainty

Assume that the contract price per share at issuance, date 0, is V_0 and that the number of shares to be issued at each date t is N_t. Let the fair value per share at date 0 be S_0. For convenience assume that the continuously compounded risk-free rate, r, is constant for all dates and the shares pay no dividends. Each future tranche is a forward contact with a value to the purchaser at date 0 equal to the difference between the forward price of the security and the issuance price, discounted to the present. The forward price is S_0e^{rt} and the value of a forward contract is:

$$N_t\left(S_0e^{rt}-V_0\right)e^{-rt}=N_t\left(S_0-V_0e^{-rt}\right) \tag{1}$$

If a contract has m future tranches, the fair value of the first payment of N_0V_0 can be expressed in terms of the fair value of the shares at date 0 as shown:

$$\begin{aligned}
&V_0N_0=S_0N_0+\left(S_0-V_0e^{-r}\right)N_1+\left(S_0-V_0e^{-2r}\right)N_2+\left(S_0-V_0e^{-3r}\right)N_3+.....+\left(S_0-V_0e^{-mr}\right)N_m\\
&S_0\sum_{i=0}^{m}N_i=V_0\sum_{i=0}^{m}N_ie^{-ir}\\
&S_0=\frac{V_0\sum_{i=0}^{m}N_ie^{-ir}}{\sum_{i=0}^{m}N_i}
\end{aligned} \tag{2}$$

Example

We illustrate the calculations involved by addressing the example used in the first paragraph. There are three tranches of 10 million shares all priced at $1.00. The firm issues tranche 1 at date 0 and will issue the other tranches 2 and 3 at dates 1 and 2. The risk-free rate of interest is 1%. Following equation (2):

$$
\begin{aligned}
\$1.00(10) &= S_0 10 + \left(S_0 - \$1.00e^{-.01}\right)10 + \left(S_0 - \$1.00e^{-.02}\right)10 \\
30S_0 &= \$1.00\left(1 + e^{-.01} + e^{-.02}\right)10 \\
S_0 &= \frac{\$1.00\left(1 + e^{-.01} + e^{-.02}\right)10}{30} = \$0.990083
\end{aligned} \tag{3}
$$

The two forward contracts have values per share of:

$$
\begin{aligned}
&\left(S_0 - V_0 e^{-.01}\right) = \left(\$0.990083 - \$1.00e^{-.01}\right) = \$0.000033 \text{ and} \\
&\left(S_0 - V_0 e^{-.02}\right) = \left(\$0.990083 - \$1.00e^{-.02}\right) = \$0.009884
\end{aligned} \tag{4}
$$

Contingent Tranche Financing

Assume that the contract price per share at issuance, date 0, is V_0 and that the number of shares to be issued at each date t is N_t. Let the fair value per share at date 0 be S_0. For convenience assume that the continuously compounded risk-free rate, r, is constant for all dates and the shares pay no dividends. The initial transaction is a sale of N_0 shares and N_t call options on the shares, with an exercise price of V_0 and time to expiration of t. The value at date 0, of a call option that expires at date t, as C_{0t}. If a contract has m future tranches, the fair value of the first payment of $N_0 V_0$ is equal to the fair value of the shares and call options at date 0:

$$
V_0 N_0 = S_0 N_0 + C_{01} N_1 + C_{02} N_2 + C_{03} N_3 + \ldots.. + C_{0m} N_m \tag{4}
$$

The values of the call options are complex functions of the value of S_0. When the issuances of future tranches is contingent on the achievement of business benchmarks, it is effective to model the shares as having two future values, one when the benchmark is achieved and one when it is not.[2] This approach permits the use of the basic principles of option pricing to value the shares and the options.

Option pricing employs two principles: no-arbitrage pricing and replicating portfolios. No-arbitrage pricing means that two things that are identical cannot sell at different prices. That is true because, if the prices are different, traders will buy at the low price and sell at the high price to make a profit. Traders will continue this activity until the two prices converge.[3] A replicating portfolio refers to the idea that a portfolio of securities can have the same payoffs as a single security. In that case, the no-arbitrage principle requires that the price of the portfolio and the price of the security it replicates must be the same.

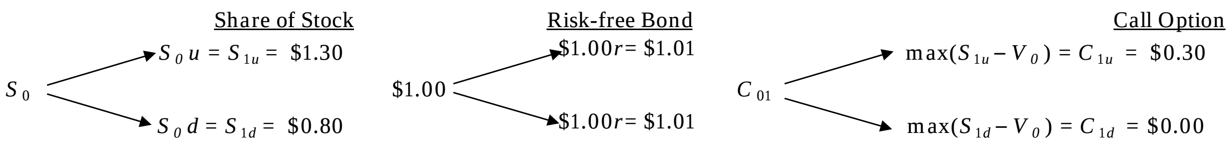
These two principles permit the pricing of a security that receives payoffs that depend on the price of a second security: They permit the pricing of a derivative based on the price of the underlying security.

[2] We consider alternative ways to model the share values below.

[3] If it costs to transact, then there can be a bid-ask spread for buying and selling prices. Similarly, if the two things are located in different places, there can be price differences based on transportation costs.

A simple example illustrates this solution technique. A firm issues one share at a price V_0 = \$1.00 and simultaneously commits to issuing one share in one year at \$1.00 if the firm achieves a business benchmark such as signing sales contracts worth \$10 million. The price per share at date 0 is S_0, and the price of the option at date 0 is C_{01}. There is a risk-free bond that pays 1% interest. In one year the price per share will either increase by the factor u to $S_0u = S_{1u}$, if the benchmark is achieved, or decrease by the factor d to $S_0d = S_{1d}$ if the benchmark is not achieved. In this case, the only inputs required to value the stock and the option are estimates of the values S_{1u} and S_{1d}. Figure 1 displays the possible payoffs in one year for each of the three securities.

Figure 1
Value Outcomes at Date 1



To value the call option payoff when the stock price appreciates, C_{01}, we create a portfolio that replicates the call's payoffs. Let the number of units of the stock in the replicating portfolio be x and the number of the units of the bond in the portfolio be y. Replication requires that:

$$1.30x + 1.01y = 0.30 \text{ and} \tag{5}$$

$$0.80x + 1.01y = 0.0 \tag{6}$$

There are two equations in two unknowns x and y. To solve the equations begin by subtracting equation (6) from equation (5):

$$0.50x = 0.30 \text{ and } x = 0.60 \tag{7}$$

Substituting equation (7) into equation (6):

$$0.80(0.60) + 1.01y = 0.0 \text{ or } y = -0.48/1.01 = -0.4752 \tag{8}$$

The replicating portfolio is a long position in 0.60 shares partially financed by borrowing \$0.4752 (a short position in the risk-free bond).

The value of the call option is equal to the cost of buying the replicating portfolio:

$$C_{01} = 0.60S_0 - \$0.4752$$

At first glance, this does not appear to be a solution, but in addition to this result we know:

$$V_0 = S_0 + C_{01} = \$1.00$$

Therefore:

$$S_0 = \$1.00 - 0.60S_0 + \$0.4752 \text{ and } S_0 = \$1.4752/1.60 = \$0.9220$$

$$C_{01} = \$1.00 - \$0.9220 = \$0.07780$$

This result highlights an important economic relationship. The analysis does not require the probabilities of the forecast outcomes, \$1.30 and \$0.80 nor the required rate of return on the share. The reason is that these values are implicit in the forecasts of future values for the stock and the original sale value, V_0. For each possible probability of \$1.30 there is an implied required rate of return and vice versa. Table 1 displays example values. As part of a review of this valuation to determine whether the values \$1.30 and \$0.80 are reasonable it is useful to review the data shown in Table 1 to determine if at least one of those pairs of values is reasonable.

Table 1
Implied Combinations of Probabilities and Required Rates of Return

Probability of \$1.30	Required Rate of Return
50%	13.9%
60%	19.3%
70%	24.7%
80%	30.1%
90%	35.6%

It is convenient to introduce a short-cut to the valuation process. It has been shown[4] that the value at date 0 of \$1.00 when the stock price appreciates and depreciates are, respectively:

$$\frac{\$1.00p}{r^*} \text{ and } \frac{\$1.00(1-p)}{r^*} \text{ where: } p = \frac{r^* - d}{u - d}\,^{5},$$

where r^* is one plus the discrete rate of interest. In this example:

$$p = \frac{1.01 - \dfrac{0.80}{0.922}}{\dfrac{1.30}{0.922} - \dfrac{0.80}{0.922}} = 0.2625 \text{ and } C_{01} = 0.2625\left(\frac{\$0.30}{1.01}\right) = \$0.078$$

[4] See for example, Hull, J. Options, Futures and Other Derivatives, 8th ed. Upper Saddle River, NJ: Prentice-Hall, 2011, p. 256.

[5] p is often referred to as “risk-neutral probability” because it falls between 0.0 and 1.0 and these two values are in the form of probability weighted payoffs discounted by the risk-free rate.

Three Extensions

a) Two Options

Consider the earlier example, but with three tranches with two business benchmarks. If the firm signs \$10 million of sales contracts in the first year, it will issue a third tranche of an equal number of shares if it signs \$15 million of sales contracts in the second year. Management estimates that the value of the stock will appreciate from \$1.30 at date 1 to \$1.75 at date two if the benchmark is achieved or depreciate to \$0.90 if it is not achieved. In this example:

$\$1.00 = S_0 + C_{01} + C_{02}$

This valuation problem requires a backward recursive solution, that is, the valuations at date 1 must be performed before the valuations at date 0. Figure 2 displays the information used to value the date 2 option at date 1 when the price appreciates, C^*_{1u}:

Figure 2
Value Outcomes at Date 2

Share of Stock		Risk-free Bond		Call Option	
Date 1	Date 2	Date 1	Date 2	Date 1	Date 2
	\$1.75		\$1.01		\$0.75
\$1.30		\$1.00		C^*_{1u}	
	\$0.90		\$1.01		\$0.00

Using the shortcut described above:

$$p_{12} = \frac{1.01 - \frac{0.90}{1.30}}{\frac{1.75}{1.30} - \frac{0.90}{1.30}} = 0.4859 \text{ and } C^*_{1u} = 0.4859\left(\frac{\$0.75}{1.01}\right) = \$0.3608$$

Figure 3 displays the information available to calculate values at date 0.

Figure 3
Value Outcomes at Date 1

Share of Stock		Risk-free Bond		Call Option	
Date 0	Date 1	Date 0	Date 1	Date 0	Date 1
	$1.30		$1.01		$0.30
S_0		$1.00		C_{01}	
	$0.80		$1.01		$0.00

Again, using the shortcut:

$$p_{01} = \frac{1.01 - \frac{0.80}{S_0}}{\frac{1.30}{S_0} - \frac{0.80}{S_0}}$$

$$S_0 = \left(\frac{p_{01}\$1.30 + (1 - p_{01})\$0.80}{1.01} \right)$$

$$C_{01} = \left(\frac{p_{01}\$0.30 + (1 - p_{01})\$0.00}{1.01} \right)$$

$$C_{02} = \frac{p_{01}C^*_{1u}}{1.01} \text{ and}$$

$$\$1.00 = S_0 + C_{01} + \frac{p_{01}C^*_{1u}}{1.01} = S_0 + C_{01} + C_{02}$$

All of the variables in the equation above depend on S_0. Applying either algebraic substitution or a numerical method we can determine the value of and all of the other variables.

$$p_{01} = 0.1809;\ S_0 = \$0.8816;\ C_{01} = \$0.0537;\ C_{02} = \$0.0646$$

b) Payoffs Defined in Terms of Firm Value Rather than Security Value

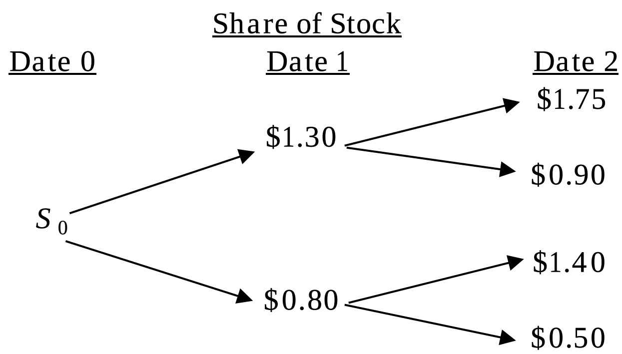
Management may provide forecasts of the value of shares being issued for the two cases where the benchmark is achieved and not achieved. It may be more likely that management would provide estimates of the value of the firm as a whole contingent on achieving or failing to achieve the benchmark. In that case, if the firm has a complex capital structure then the valuation of the tranche option will require backsolving for the share values in the two valuation scenarios and then proceeding as described above.

This approach is also feasible for the case in which management describes either the future value of the share or of the firm as a pair of distributions with different expected values and volatilities. In this case, the options can be valued using the Black-Scholes-Merton formula[6] and the analysis proceeds as described above.

c) More Complex Value Branching

Figure 4 describes a slightly different set of facts. In this case, the third tranche is not contingent on the second tranche being issued and the third tranche option could be in the money even if the first tranche option is not.

Figure 4
Value Outcomes When Tranche 3 is not Contingent on Tranche 2



With these scenarios, when the firm achieves both the first and second benchmarks the security price appreciates to $1.30 and then to $1.75 and both of the tranche options are in the money. When the firm does not achieve the first benchmark but does achieve the second benchmark the security price depreciates to $0.80 and then appreciates to $1.40 and only the third-tranche option is in the money. The valuation of the share and the call options proceeds exactly as previously described with the addition of the in-the-money call option when the share price is $1.40. This example indicates that the method is feasible with more complex price paths. However, this option pricing methodology relies on the binomial feature, namely that at each node in the price tree, there are at most two future prices.

Applying the same valuation technique, in this example:

$$S_0 = \$0.8280;\ C_{01} = \$0.0216; C_{02} = \$0.1504$$

Conclusions

Multi-tranche equity financing creates derivatives if subsequent tranches are either certain or contingent on future events. In this article we have discussed how to bifurcate the value of the issued equity from the derivatives using relatively modest level input from management that is likely to be available.

[6] See Hull, p. 314.

References

Hull, J. Options, Futures and Other Derivatives, 8th ed. Upper Saddle River, NJ: Prentice-Hall, 2011.