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Mean Reversion Models and No-Arbitrage Prices

by

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Motivation

I recently valued a complex derivative where WTI oil was the underlying price. The valuation required a Monte Carlo simulation. When the valuation was reviewed by a major accounting firm, the values produced by their specialists deviated from mine more than expected. After a cooperative exchange, I identified the primary source of the difference. The reviewers' model produced average future spot prices that were not equal to the market forward price. This violates the "no-arbitrage" requirement of pricing models. The source of the error was the omission of "drift-adjustment" terms.

To avoid misunderstanding, let me use a familiar example to indicate what was missing. In a risk-neutral simulation of stock prices that follow geometric Brownian motion (GBM), the total drift in the stock price is $\left(r - \frac{\sigma^2}{2}\right)\Delta t$, where r is the risk-free interest rate, $\left(-\frac{\sigma^2}{2}\right)$, is the drift-adjustment term (*DAT*) and Δt is a unit of time.

The application of Ito's lemma gives rise to the *DAT* in mean-reversion models, just as it does in GBM models. This has been recognized in the continuous-time literature on mean-reversion models.[1] However, I wanted to refer the reviewers to a published reference that would explain how to calculate the *DAT*s required in a discrete time model. If such a reference exists, I was not able to find it. Therefore, I prepared this note.

I begin by presenting a derivation of the GBM *DAT*, $\left(-\frac{\sigma^2}{2}\right)$. I think that will help the reader follow the more complex deviation for the mean-reversion of *DAT*s. After completing the derivation of *DAT*s for a Monte Carlo simulation, I wondered how these results would apply to building a lattice of prices for a mean-reverting process. I reviewed Hull's[2] excellent textbook on how to derive a trinomial lattice of prices for a mean reverting process. His method involves a search process to identify the *DAT*s. The derivation of the *DAT*s for the Monte Carlo implementation provides the basis for the analytical calculation of *DAT*s required by lattices. I use Hull's example to demonstrate how to use the analytical solution and avoid the search process.

Derivation of the *DAT*

1. Geometric Brownian Motion

We assume that a stock price at date t, S_t, follows a stochastic process with a constant drift rate of r and a constant volatility of σ:

$$dS_t = rS_t dt + \sigma S_t dz$$

where dz is a Wiener process. For this stock price process, we know from the application of Ito's lemma that the rate of return on S_t,

$x_t = \ln S_t$, follows the process:

[1] Eduardo S. Schwartz. "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging", *Journal of Finance*, Vol. 52, No. 3, p. 926. (I want to thank Andrew Lyasoff for suggesting this reference.)
[2] John C. Hull. *Options Futures and Other Derivatives*, *8th Edition*, 2012, Prentice-Hall, New York, N. Y.

$dx_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dz.$

The *DAT* is equal to one-half the variance,

$DAT = \left(-\frac{\sigma^2}{2}\right).$

The stock price rate of return is normally distributed with a mean and standard deviation over the time interval *T* equal to $\left(r - \frac{\sigma^2}{2}\right) T$ and $\sigma\sqrt{T}$, respectively.

The realtionship,

$x_t + dx_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dz,$

can be written in discrete time as:

$x_{t+\Delta t} = x_t + \Delta x_t = x_t + \left(r - \frac{\sigma^2}{2}\right) \Delta t + \sigma\sqrt{\Delta t} z_{t+\Delta t},$

where z_t is a unit normal random variate. Without loss of generality we can set $t = 0$ and $\Delta t = 1$. Then:

$x_1 = x_0 + r + DAT_1 + \sigma z_1 = x_0 + \left(r - \frac{\sigma^2}{2}\right) + \sigma z_1,$

$x_2 = x_1 + r + DAT_2 + \sigma z_2 = x_0 + \left(r - \frac{\sigma^2}{2}\right) + r + DAT_2 + \sigma(z_1 + z_2).$

The variance of x_2 is $2\sigma^2$. Therefore, by Ito's lemma:

$DAT_1 + DAT_2 = -\frac{2\sigma^2}{2}$ and $DAT_2 = \left(-\frac{\sigma^2}{2}\right).$

We have the general relationship that x_n is normally distributed with mean and volatility equal to $n\left(r - \frac{\sigma^2}{2}\right)$ and $\sigma\sqrt{n}$, and

$DAT_i = DAT_j = \left(-\frac{\sigma^2}{2}\right)$

2. Mean Reversion

In this case, a commodity price at date t, S_t, follows a mean-reverting stocastic process with a time-varying expected value, a rate of reversion, θ, to the expected value of μ_t and a constant volatility of σ:

$dS_t = \theta(\mu_t - x_t)S_t dt + r_t S_t dt + \sigma S_t dz.$

We include $r_t S_t$ to capture the time-varying drift in prices that is typical of commodities. For this price process, we know from the application of Ito's lemma that the rate of return on S_t, $x_t = \ln S_t$, follows the process:

$dx_t = \theta(\mu_t - x_t)dt + r_t dt + DAT_t dt + \sigma dz.$

We can also write:

$x_t + dx_t = \theta\mu_t dt + x_t(1-\theta)dt + r_t dt + DAT_t dt + \sigma dz$

In discrete time this is:

$x_t + \Delta x_{t+1} = (1-k)\mu_t + kx_t + r_{t+1} + DAT_{t+1} + b\sigma\sqrt{\Delta t}z_{t+1},$

where

$k = e^{-\theta\Delta t}$ *and* $b = \left(\frac{(1-e^{-2\theta\Delta t})}{2\theta}\right)^{0.5}.$

Without loss of generality we can set $t = 0$ and Δt equal to 1.0. Then,

$x_1 = (1-k)\mu_0 + kx_0 + r_1 + DAT_1 + b\sigma z_1.$

The variance of x_1 is $b^2\sigma^2$. Therefore, by Ito's lemma, $DAT_1 = -\frac{b^2\sigma^2}{2}$.

Similarly,

$$
\begin{aligned}
x_2 &= (1-k)\mu_1 + kx_1 + r_2 + DAT_2 + b\sigma z_2, \text{ or} \\
&= (1-k)\mu_1 + k\big((1-k)\mu_0 + kx_0 + r_1 + DAT_1 + b\sigma z_1\big) + r_2 + DAT_2 + b\sigma z_2 \\
&= (1-k)\mu_1 + k(1-k)\mu_0 + k^2x_0 + kr_1 + kDAT_1 + r_2 + DAT_2 + b\sigma(kz_1 + z_2).
\end{aligned}
$$

The variance of x_2 is $b^2\sigma^2(k^2+1)$. Therefore, by Ito's lemma,

$kDAT_1 + DAT_2 = -\frac{b^2\sigma^2}{2}(k^2+1).$

and

$$DAT_2 = -\frac{b^2\sigma^2}{2}(k^2+1) - kDAT_1 = -\frac{b^2\sigma^2}{2}(k^2+1) - k\frac{b^2\sigma^2}{2} = -\frac{b^2\sigma^2}{2}(k^2-k+1)$$

Similarly,

$$
\begin{aligned}
x_3 &= (1-k)\mu_2 + kx_2 + r_3 + DAT_3 + b\sigma z_3 \\
x_3 &= (1-k)\mu_2 + k(1-k)\mu_1 + k^2(1-k)\mu_0 + k^3x_0 + k^2r_1 + kr_2 + r_3 + k^2DAT_1 + kDAT_2 \\
&\quad + DAT_3 + b\sigma(k^2z_1 + kz_2 + z_3)
\end{aligned}
$$

The variance of x_3 is $b^2\sigma^2(k^4+k^2+1)$. Therefore, by Ito's lemma,

$k^2DAT_1 + kDAT_2 + DAT_3 = -\frac{b^2\sigma^2}{2}(k^4+k^2+1)$

and

$$DAT_3 = -\frac{b^2\sigma^2}{2}(k^4 + k^2 + 1) - k^2 DAT_1 - kDAT_2$$
$$= -\frac{b^2\sigma^2}{2}(k^4 + k^2 + 1) + \frac{b^2\sigma^2}{2}(k^3 + k)$$
$$= -\frac{b^2\sigma^2}{2}(k^4 - k^3 + k^2 - k + 1)$$

In general, the variance of x_n is $b^2\sigma^2\left(k^{(n-1)^2} + k^{(n-2)^2} + \cdots + k^2 + 1\right)$ and

$$DAT_n = -\frac{b^2\sigma^2}{2}\left(k^{(n-1)^2} - k^{(n-1)^2-1} + k^{(n-2)^2} - k^{(n-2)^2-1} + \cdots + k^2 - k + 1\right).$$

As n becomes large DAT_n approaches $-\frac{b^2\sigma^2}{2}\left(\frac{1}{1+k}\right)$.

A Monte Carlo Illustration

The random prices in the Monte Carlo simulation are:

$$S_{t+1} = \exp[x_{t+1}] = \exp[(1-k)\mu_t + kx_t + r_{t+1} + DAT_{t+1} + b\sigma z_{t+1}].$$

Suppose that the current price of oil and its anticipated price, S_0 and μ_o are both \$60. The futures prices of oil, F_t, 1, 2 and 3 years hence are \$65, \$62 and \$58. The volatility is 20% and the rate of mean reversion is 10%. Then we have this table.

σ	20%			
Δt	1.0			
θ	10%			
b	0.952			
k	0.905			
$(1 - k)$	0.095			
t	0.0	1.0	2.0	3.0
F_t	\$60.00	\$65.00	\$62.00	\$68.00
μ_t	4.0943	4.1744	4.1271	4.2195
$r_t = \ln(S_{t+1}/S_t)$		0.080	-0.0473	0.0924
DAT_t		-0.0190	-0.0174	-0.0147

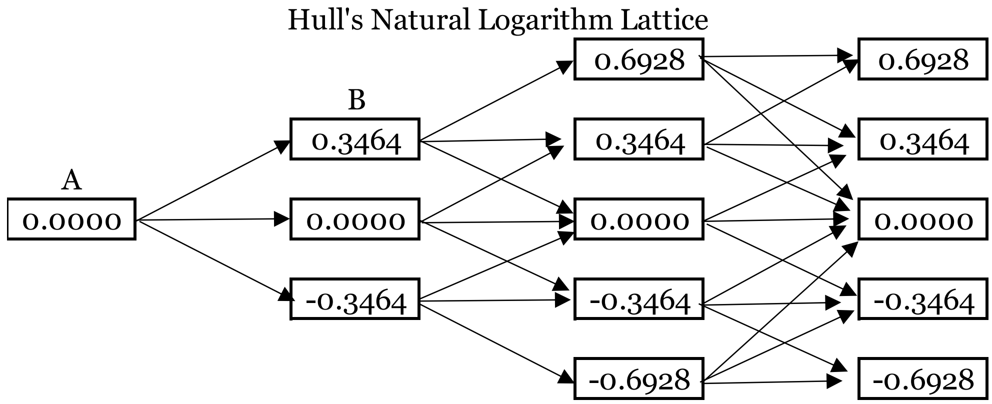
$$S_1 = \exp[(0.095)(4.0943) + (0.905)(4.0943) + 0.08 - 0.0190 + (0.952)(0.20)z_{t+1}]$$

$$S_2 = \exp[(0.095)(4.1744) + (0.905)(\ln S_1) - 0.047 - 0.0174 + (0.952)(0.20)z_{t+2}]$$

$$S_3 = \exp[(0.095)(4.1271) + (0.905)(\ln S_2) + 0.0924 - 0.0147 + (0.952)(0.20)z_{t+3}]$$

A Trinomial Lattice Illustration

Hull (p. 754- 55) illustrates the creation of a mean-reversion trinomial price lattice by starting with a natural logarithm lattice that is symmetric around 0.0 with nodes and probabilities that produce a desired spot volatility, 20%, and a desired rate of mean reversion, 10%. The time step is one year. Consider nodes A and B in his lattice.



At nodes A and B, we have the following probability distribution of outcomes, expected values and volatilities.

Node A		Node B	
Pr	Outcome	Pr	Outcome
0.1667	0.3464	0.1217	0.6928
0.6667	0	0.6567	0.3464
0.1667	-0.3464	0.2217	0.0000

Expected Value	0.0000	0.3118
Volatility	20.00%	20.00%

At node A, the initial value and the usual value are both 0.0. Therefore, the expected future value is also 0.0 and there is no mean-reversion effect. The volatility is 20%. At node B the initial value is 0.3464 and the usual value is 0.0. Through the choice of probabilities, the mean-reversion effect creates an expected future value, 0.3118, which is 90% of the initial value of 0.3464: the rate of mean-reversion 10%. Note that 0.90 is the equivalent of the variable k in the earlier discussion.

Hull's price lattice is built on the logarithm lattice by adding a constant logarithm at each date such that the expected value of the price at each date is equal to the forward price for that date. In his example, the forward price at date 1 is 22. To find the value of the logarithm to add, Hull solves the following equation by a search process:

$$22.00 = 0.1667(\exp[0.3464 + \alpha_1]) + 0.6667(\exp[\alpha_1]) + 0.1667(\exp[-0.3464 + \alpha_1]).$$

Hull's method of building the price lattice requires solving similar but more complex equations at each date.

Consistent with the Monte Carlo simulation analysis, there is an analytical definition that eliminates the searches. Consider α_1, which is 3.07104. That value is equal to ln(22.00) - 0.02, where 0.02 is equal to -0.5(20%²), the one-period *DAT*. In general, Hull's alphas are equal to the natural logarithm of the forward price reduced by one-half the variance of their distribution, what we call a total drift adjustment term. As we showed earlier, the variance of the price at date 2 is:

$(20\%^2)(k^2 + 1) = (20\%^2)(0.90^2 + 1)$ and for the price at date 3 it is:

$(20\%^2)(k^4 + k^2 + 1) = (20\%^2)(0.90^4 + 0.90^2 + 1)$, etc.

The table below summarizes the results for this example and provides a comparison of the results of Hull's search solution and my analytical solution.

Date	1	2	3
Forward price	22.00	23.00	24.00
ln of forward price	3.09104	3.13549	3.17805
Hull's α	3.07104	3.09930	3.12881
Hull's total drift adjustment	-0.02000	-0.03619	-0.04924
Hull's model price	22.00000	23.00000	24.00000
Grant's total drift adjustment	-0.02000	-0.03620	-0.04932
Grant's model price	21.99999	22.99996	23.99810

As this example indicates, we can use this analytical result to effectively build trinomial mean-reversion price lattices.[3]

[3] $-0.03620 = -0.5(20\%^2)(0.90^2 + 1)$

$-0.04932 = -0.5(20\%^2)(0.90^4 + 0.90^2 + 1)$